Linear Algebra, Winter 2022

List 4

Determinants, inverses, systems

74. (a) Calculate $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^2 = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$.

    (b) Calculate $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^4$. (b) Calculate $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^5$. (c) Calculate $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^{10}$.

    ☆(d) Looking at $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^2$, $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^3$, etc., guess a formula for $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^n$.

75. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be $T(x,y) = (2x+y, -x+2y)$ and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be $S(x,y) = (3y, -x)$. Compute $S(T(3,1))$.

76. For T and S from Problem 75, give the matrix for T, the matrix for S, and matrix for $S(T(x,y))$.

77. (a) Is $\begin{bmatrix}2 & 5\\9 & -2\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}$ parallel to $\begin{bmatrix}4\\4\end{bmatrix}$?

    (b) Is $\begin{bmatrix}2 & 5\\9 & -2\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}$ parallel to $\begin{bmatrix}2\\5\end{bmatrix}$?

    (c) Is $\begin{bmatrix}2 & 5\\9 & -2\end{bmatrix}\begin{bmatrix}2\\9\end{bmatrix}$ parallel to $\begin{bmatrix}2\\9\end{bmatrix}$?

    (d) Is $\begin{bmatrix}2 & 5\\9 & -2\end{bmatrix}\begin{bmatrix}5\\-9\end{bmatrix}$ parallel to $\begin{bmatrix}5\\-9\end{bmatrix}$?

    (e) Is $\begin{bmatrix}2 & 5\\9 & -2\end{bmatrix}\begin{bmatrix}2\\-9\end{bmatrix}$ parallel to $\begin{bmatrix}2\\-9\end{bmatrix}$?

    (f) Is $\begin{bmatrix}2 & 5\\9 & -2\end{bmatrix}\begin{bmatrix}2\\-2\end{bmatrix}$ parallel to $\begin{bmatrix}2\\-2\end{bmatrix}$?

> The **determinant** of a square matrix A is written as $\det(A)$. For a 2×2 matrix,
>
> $$\det\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = ad - bc.$$
>
> For larger matrices, the formula is more difficult. Properties include
>
> $$\det(AB) = \det(A)\cdot\det(B) \qquad\text{and}\qquad \det(sA) = s^n\det(A)$$
>
> if A is an $n\times n$ matrix. Geometrically, $|\det(A)|$ is the volume of the parallelepiped (in 2D, area of the parallelogram) whose edges, as vectors, are the columns of A.

78. Compute the determinants of the following matrices.

    (a) $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

    (b) $\begin{bmatrix}3 & 5\\5 & 9\end{bmatrix}$

    (c) $\begin{bmatrix}8 & 4\\6 & 3\end{bmatrix}$

    (d) $\begin{bmatrix}3 & -1\\2 & 5\end{bmatrix}$

    (e) $\begin{bmatrix}3 & b\\2 & 5\end{bmatrix}$

    (f) $\begin{bmatrix}1 & 3 & 5\\2 & 4 & 9\\5 & 6 & 8\end{bmatrix}$

    ☆(g) $\begin{bmatrix}3 & 1 & 3 & 3\\1 & 2 & 4 & 0\\2 & 0 & 2 & 2\\4 & 2 & 1 & 3\end{bmatrix}$

79. If M is a 5×5 matrix with $\det(M) = 2$ compute $\det(2M)$ and $\det(-3M^2)$.

The $\boldsymbol{n \times n}$ **identity matrix** is the matrix I (also written I_n or $I_{n\times n}$) such that

$$IM = MI = M$$

for any $n \times n$ matrix M. It has 1 along the main diagonal and 0 everywhere else.

80. (a) Multiply $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 30 & -1 & 10 \\ \frac{121}{2} & -2 & 20 \end{bmatrix}$.

(b) Multiply $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 30 & -1 & 10 \\ \frac{121}{2} & -2 & 20 \end{bmatrix} \begin{bmatrix} 6 & 4 & 9 \\ 0 & -3 & -2 \\ 4 & 6 & -4 \end{bmatrix}$.

The **inverse matrix** of a square matrix M is written M^{-1} (spoken as "M inverse") and it is the unique matrix for which $M^{-1}\,M = I$. An inverse matrix exists if and only if $\det(M) \neq 0$. For a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For larger matrices, the formula is much more difficult but includes $\frac{1}{\det(M)}$.

81. Find $\begin{bmatrix} 5 & 4 \\ 1 & -2 \end{bmatrix}^{-1}$ and $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix}^{-1}$.

82. Find the inverses of the matrices from Task 78, if they exist. (☆(f), ☆(g))

83. Find the matrix M from Task 59.

84. For each of the following, does an inverse matrix exist?

(a) Matrix A, a 3×3 matrix with $\det(A) = 3$.

(b) Matrix B, a 3×5 matrix where every number in the matrix is 1.

(c) Matrix C, a 4×4 matrix where every number in the matrix is 1.

(d) Matrix D, a 4×4 matrix where every number in the matrix is 0.

(e) Matrix E, a 5×5 matrix with $\det(D) = -1$.

(f) Matrix F, a 7×7 matrix with $\det(E) = 0$.

(g) Matrix G, a 2×2 matrix with $a_{ij} = i + j$.

85. For what values of p are each of the following matrices invertible? Give a formula for the inverse of each matrix.

(a) $\begin{bmatrix} 1 & 2 \\ p & p^3 \end{bmatrix}$ (b) $\begin{bmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - pI_{2\times 2}$

86. Let $f : \mathbb{R}^2 \to R^2$ be given by $f(x, y) = (9x - 11y, 4x - 5y)$. Find values of a and b such that $f(a, b) = (7, 2)$.

87. Let $f : \mathbb{R}^2 \to R^2$ be given by $f(x, y) = (5x, 10x + y)$. Give a formula for $f^{-1}(x, y)$, that is, the function for which $f^{-1}(f(x, y)) = (x, y)$.

A collection of vectors is called **linearly dependent** if one of the vectors is a linear combination of the others. Otherwise it is **linearly independent**.

Alternate definition: a collection $\{\vec{v_1}, ..., \vec{v_n}\}$ is linearly dependent if there exist scalars $s_1, ..., s_n$ not *all* zero (but some may be zero) such that $s_1\vec{v_1} + \cdots + s_n\vec{v_n} = \vec{0}$.

88. For the vectors

$$\vec{v_1} = [2, 9, -6], \qquad \vec{v_2} = [4, 2, -6], \qquad \vec{v_3} = [0, -8, 3],$$

is the collection $\{\vec{v_1},\ \vec{v_2},\ \vec{v_3}\}$ linearly dependent or independent?
Hint: See Task 28 from List 1.

89. Determine whether each collection is linearly dependent or independent.
   (a) $\{\hat{\imath},\ \hat{\jmath}\}$
   (b) $\{\hat{\imath},\ \hat{\jmath},\ \hat{k}\}$
   (c) $\{\hat{\imath},\ \hat{\jmath},\ \vec{0}\}$
   (d) $\left\{ \begin{bmatrix} 44 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ 21 \end{bmatrix}, \begin{bmatrix} 21 \\ 49 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \end{bmatrix} \right\}$
   (e) $\left\{ \begin{bmatrix} 15 \\ -35 \end{bmatrix} \right\}$
   (f) $\left\{ \begin{bmatrix} 15 \\ -35 \end{bmatrix}, \begin{bmatrix} -9 \\ 21 \end{bmatrix} \right\}$
   (g) $\left\{ \begin{bmatrix} 5 \\ 50 \\ -100 \end{bmatrix}, \begin{bmatrix} 0 \\ 100 \\ 200 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

90. For each, state whether the collection must be linear dependent, must be linear independent, or that there is not enough information to know this.
   (a) A collection of 10 vectors each of dimension 3.
   (b) A collection of 3 vectors each of dimension 10.
   (c) A collection of 20 vectors each of dimension 10.
   (d) A collection of 3 vectors, each of dimension 3, that includes two parallel vectors.
   (e) A collection of 3 vectors, each of dimension 3, that includes two perpendicular vectors.
   (f) A collection of 3 vectors, each of dimension 3, where each vector is perpendicular to the other two.
   (g) A collection of 3 vectors, each of dimension 3, that includes the zero vector.

91. Is the collection of vectors

$$\left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

linearly dependent or linearly independent?

92. Give an example of a vector $\vec{u}$ for which

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \vec{u} \right\}$$

is linearly independent.

93. Write $[8, 3, 1]$ as a linear combination of $[1, 0, 1]$ and $[2, 0, 3]$ or explain why it is impossible to do so.

94. Write $[5, 5, 1]$ as a linear combination of $[1, 1, 1]$ and $[0, 0, 8]$ or explain why it is impossible to do so.

95. (a) Is the collection $\{[-1, 8, 8]\}$ linearly independent?
    (b) Is the collection $\{[-1, 8, 8], [5, 0, 0]\}$ linearly independent?
    (c) Is the collection $\{[-1, 8, 8], [5, 0, 0], [3, 1, 3]\}$ linearly independent?
    (d) Is the collection $\{[-1, 8, 8], [5, 0, 0], [3, 1, 3], [3, -4, 4]\}$ linearly independent?

96. (a) Is the collection $\{[0, 2, 5]\}$ linearly independent?
    (b) Is the collection $\{[0, 2, 5], [1, 1, -4]\}$ linearly independent?
    (c) Is the collection $\{[0, 2, 5], [1, 1, -4], [2, 4, -3]\}$ linearly independent?
    (d) Is the collection $\{[0, 2, 5], [1, 1, -4], [2, 4, -3], [2, 8, 7]\}$ linearly independent?

The **rank** of a matrix is the maximum number of linearly independent rows in the matrix. This is also the maximum number of linearly independent columns.

97. Give the rank of the following matrices:

(a) $\begin{bmatrix} -1 & 8 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 8 & 8 \\ 5 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 8 & 8 \\ 5 & 0 & 0 \\ 3 & 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 8 & 8 \\ 5 & 0 & 0 \\ 3 & 1 & 3 \\ 3 & -4 & 4 \end{bmatrix}$

98. Give the rank of the following matrices:

(a) $\begin{bmatrix} 0 & 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & 5 \\ 1 & 1 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 & 5 \\ 1 & 1 & -4 \\ 2 & 4 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 & 5 \\ 1 & 1 & -4 \\ 2 & 4 & -3 \\ 2 & 8 & 7 \end{bmatrix}$

99. Explain why $\text{rank}(M) = 3$ for the matrix $M = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 5 \\ 6 & 3 & 2 \end{bmatrix}$.

☆100. For which values of p does $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 5 \\ 6 & 3 & p \end{bmatrix}$ have rank 2?

For which values does it have rank 3? Rank 1?

101. Find $\text{rank}(A)$ and $\det(A)$ for the matrix $A = \begin{bmatrix} 4 & 8 & 1 \\ 7 & 14 & 6 \\ 1 & 2 & 6 \end{bmatrix}$ without a calculator.

102. The rank of $\left[\begin{smallmatrix} 4 & 1 & 5 \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ is 2 and the rank of $\left[\begin{smallmatrix} 4 & 1 & 5 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & -1 & 0 & 7 \end{smallmatrix}\right]$ is 3.

How many solutions does the system $\begin{cases} 4x + y + 5z = 1 \\ 2x + \phantom{y + {}} 2z = 2 \\ x - y \phantom{{}+ 5z} = 7 \end{cases}$ have?

103. The rank of $\left[\begin{smallmatrix} 4 & 1 & 5 \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ is 2 and the rank of $\left[\begin{smallmatrix} 4 & 1 & 5 & 6 \\ 2 & 0 & 2 & 4 \\ 1 & -1 & 0 & 4 \end{smallmatrix}\right]$ is also 2.

How many solutions does the system $\begin{cases} 4x + y + 5z = 6 \\ 2x + \phantom{y + {}} 2z = 4 \\ x - y \phantom{{}+ 5z} = 4 \end{cases}$ have?

104. The rank of $\left[\begin{smallmatrix} 7 & 2 & 5 \\ 9 & 0 & 3 \\ 3 & -1 & 0 \end{smallmatrix}\right]$ is 3 and the rank of $\left[\begin{smallmatrix} 7 & 2 & 5 & -1 \\ 9 & 0 & 3 & 0 \\ 3 & -1 & 0 & 5 \end{smallmatrix}\right]$ is also 3.

How many solutions does the system $\begin{cases} 7x + 2y + 5z = -1 \\ 9x + \phantom{2y + {}} 3z = 0 \\ 3x - y \phantom{{}+ 5z} = 5 \end{cases}$ have?

105. If the numbers a, b, c, d are such that $(x, y) = (9, 1)$ is a solution to $\begin{cases} ax + by = 2 \\ cx + dy = 3 \end{cases}$ but the system $\begin{cases} ax + by = 4 \\ cx + dy = 9 \end{cases}$ has no solutions, what is the rank of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?